A LOWER BOUND FOR THE AMPLITUDE OF TRAVELING WAVES OF SUSPENSION BRIDGES

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ABSTRACT. We obtain a lower bound for the amplitude of nonzero homoclinic traveling wave solutions of the McKenna–Walter suspension bridge model. As a consequence of our lower bound, all nonzero homoclinic traveling waves become unbounded as their speed of propagation goes to zero (in accordance with numerical observations).

We study traveling wave solutions of the McKenna–Walter suspension bridge model

$$u_{tt} + u_{xxxx} + f(u) = 0 \tag{1}$$

introduced in [8]. Two of the most standard choices for the nonlinear term f are

$$f(u) = \max(u, -1), \qquad f(u) = e^u - 1. \tag{2}$$

The piecewise linear choice was the original choice in [8] and stems from the fact that the cables of a suspension bridge will resist movement only in one direction. The exponential choice was introduced in [2] as a smooth version of the piecewise linear one which behaves the same way as $u \to -\infty$ and also near $u = 0$. The smooth version is more suitable when it comes to numerics and also seems more appealing to engineers [4].

Numerical results for (1) go back to McKenna and Walter [8] who studied traveling wave solutions. Those are solutions of the form $u = u(x - ct)$, so they satisfy the ODE

$$u'''' + c^2 u'' + f(u) = 0. \tag{3}$$

Based on the numerical results of [8] for the piecewise linear model and those of [2, 3] for the exponential one, traveling waves become unbounded as $c \to 0$. A rigorous proof of this observation was given by Lazer and McKenna [6], but it only applied to the piecewise linear model and not the exponential one.

In this paper, we give a simple proof that applies to any nonlinear term f such that

(A1) f is locally Lipschitz continuous with $uf(u) > 0$ for all $u \neq 0$, and
(A2) f is differentiable at the origin with $f'(0) > 0$.

Clearly, these assumptions hold for both nonlinearities in (2). To show that traveling waves become unbounded as their speed goes to zero, we actually prove a lower bound for their amplitude, which is of independent interest by itself. In what follows, we shall only focus on homoclinic solutions, namely those which vanish at $\pm\infty$.

Theorem 1. *Assume (A1)-(A2) and that $0 < c^4 < 4f'(0)$. If u is a nonzero homoclinic solution of equation (3), then $||u||_\infty \geq L(f, c)$, where*

$$L(f, c) = \sup\left\{\delta > 0 : \frac{f(u)}{u} > \frac{c^4}{4} \text{ whenever } 0 \neq |u| < \delta\right\}. \tag{4}$$

We remark that the lower bound $L(f, c)$ is well-defined because

$$\lim_{u \to 0} \frac{f(u)}{u} = f'(0) > \frac{c^4}{4}.$$

Moreover, $L(f, c) \to \infty$ as $c \to 0$ by (A1), so nonzero homoclinic solutions of (3) become unbounded as $c \to 0$. Our assumption that $c^4 < 4f'(0)$ is natural because the eigenvalues of the linearized problem become purely imaginary when $c^4 \geq 4f'(0)$, so one does not expect homoclinic solutions in that case.

The existence of homoclinic solutions of (3) has been studied by several authors. We refer the reader to [2] for the piecewise linear case, [5] for more general nonlinearities that grow polynomially and [9] for the exponential case. The authors of [1] studied the qualitative properties of solutions for nonlinearities that satisfy (A1). There is also an existence result by Levandosky [7] when $f(u) = u - |u|^{p-1}u$, but this case is somewhat different because it does not satisfy (A1) and the solutions remain bounded as $c \to 0$.

To prove Theorem 1, we shall need to use the following facts from [1]. We only give the proof of the last two parts and refer the reader to [1, Proposition 11] for the first.

Lemma 2. *Suppose u is a nonzero homoclinic solution of equation (3), namely a solution of equation (3) that vanishes at $\pm\infty$.*

(a) *Assume f is continuous with $f(0) = 0$. Then u', u'', u''' must also vanish at $\pm\infty$.*
(b) *Assume (A1). Then $u(s)$ must change sign infinitely many times as $s \to \pm\infty$.*
(c) *Assume (A1)-(A2) and that $0 < c^4 < 4f'(0)$. Then $u \in H^2$.*

Proof. To prove (b), we note that $u(s)$ is a solution of (3) if and only if $u(-s)$ is. Thus, it suffices to show that $u(s)$ changes sign infinitely many times as $s \to \infty$. Suppose $u(s)$ is eventually non-negative, the other case being similar. Then $w(s) = u''(s) + c^2 u(s)$ satisfies

$$w''(s) = -f(u(s)) \leq 0$$

for large enough s, so $w(s)$ is eventually concave. Since $w(s)$ goes to zero by part (a), this implies that $w(s)$ is eventually non-positive. Using this fact, we now get

$$u''(s) = w(s) - c^2 u(s) \leq 0$$

for large enough s, so $u(s)$ is eventually concave. As before, this implies $u(s)$ is eventually non-positive, so we must actually have $u \equiv 0$, a contradiction.

To prove (c), we consider the function

$$H(s) = u'(s)u''(s) - u(s)u'''(s) - c^2 u(s)u'(s). \tag{5}$$

We note that $H(s)$ is bounded by part (a) and that a short computation gives

$$H(s_2) - H(s_1) = \int_{s_1}^{s_2} [u''(s)^2 - c^2 u'(s)^2 + u(s)f(u(s))] \, ds \tag{6}$$

for all $s_1 < s_2$. Now, fix some $0 < \varepsilon < f'(0) - \frac{c^4}{4}$ and let $s_0 \in \mathbb{R}$ be such that

$$|s| \geq s_0 \quad \implies \quad \frac{f(u(s))}{u(s)} \geq f'(0) - \varepsilon.$$

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Recalling part (b), suppose s_1, s_2 are any two roots of $u(s)$ for which $|s_1|, |s_2| > |s_0|$. Then we may combine the last two equations to find that

$$H(s_2) - H(s_1) \geq \int_{s_1}^{s_2} \left[u''(s)^2 - c^2 u'(s)^2 + (f'(0) - \varepsilon) u(s)^2 \right] ds \tag{7}$$

$$\geq -2\sqrt{f'(0) - \varepsilon} \int_{s_1}^{s_2} u''(s) u(s)\, ds - c^2 \int_{s_1}^{s_2} u'(s)^2\, ds$$

$$= \alpha \int_{s_1}^{s_2} u'(s)^2\, ds,$$

where $\alpha = 2\sqrt{f'(0) - \varepsilon} - c^2 > 0$. Since $H(s)$ is bounded by above, this implies $u' \in L^2$. Using this fact and the inequality (7), we conclude that $u \in H^2$. $\blacksquare$

Proof of Theorem 1. We multiply equation (3) by u and integrate by parts to get

$$\int_{-\infty}^{\infty} u''(s)^2\, ds - c^2 \int_{-\infty}^{\infty} u'(s)^2\, ds + \int_{-\infty}^{\infty} u f(u)\, ds = 0.$$

Using the Fourier transform and a trivial estimate, we conclude that

$$\int_{-\infty}^{\infty} u f(u)\, ds = c^2 \int_{-\infty}^{\infty} u'(s)^2\, ds - \int_{-\infty}^{\infty} u''(s)^2\, ds \leq \frac{c^4}{4} \int_{-\infty}^{\infty} u(s)^2\, ds. \tag{8}$$

Since $\lim_{u \to 0} \frac{f(u)}{u} = f'(0) > \frac{c^4}{4}$, we can always find some $\delta > 0$ such that

$$0 \neq |u| < \delta \quad \Longrightarrow \quad u f(u) > \frac{c^4 u^2}{4}.$$

Suppose $\delta > 0$ is any number with this property. Given a nonzero homoclinic solution u for which $||u||_\infty < \delta$, we can then use the last equation to get

$$\int_{-\infty}^{\infty} u f(u)\, ds > \frac{c^4}{4} \int_{-\infty}^{\infty} u(s)^2\, ds,$$

contrary to (8). This means that $||u||_\infty \geq \delta$ for each nonzero homoclinic solution and each such δ, so the result follows. $\blacksquare$

The following corollary is a trivial consequence of our estimate (8). This result refines [1, Theorem 13i] which imposes the stronger assumption $\frac{f(u)}{u} \geq f'(0)$.

Corollary 3. *Assume (A1)-(A2) and that $0 < c^4 < 4f'(0)$. If $\frac{f(u)}{u} > \frac{c^4}{4}$ for all $u \neq 0$, then equation (3) has no nonzero homoclinic solutions.*

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